

Premetric electrodynamics*

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Abstract

Classical electrodynamics can be divided into two parts. In the first one, with the use of a plenty of directed quantities, namely multivectors and differential forms, no scalar product is necessary. It is called premetric electrodynamics. In this part, principal laws of the theory can be tackled, among them the two observer-independent Maxwell's equations. The second part concerns specific media and requires establishing of a scalar product and, consequently a metric. We present an axiomatic approach to electrodynamics in which the metric enters as late as possible. Also a line of research is mentioned in which the notion of non-inertial observer is studied and its influence on observer-dependent Maxwell equations.

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1 Introduction

In the last decades, a way of presenting electrodynamics has been proposed based on a broad use of differential forms; see Refs. [1]–[11]. Within this approach, a question was discussed whether electrodynamics can be formulated in such a way that the metric of space-time doesn't enter the fundamental laws of it which is called *premetric electrodynamics*, see the papers [11]–[18]. Related developments can be also found in the books [19]–[21]. A crowning achievement of this approach is the book by Hehl and Obukhov [22].

I consider useful to present this study to the Clifford Algebra community. It is built deductively, i.e. in the form of axioms: conservation of electric charge, magnetic flux and energy-momentum. At the end, the metric is introduced by the constitutive relations between the electromagnetic field quantities, namely D , H , called *excitations*, united into the four-dimensional quantity G , and B , E , called *field strengths*, united into four-dimensional quantity F .

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Each axiom introduces a physical quantity as a four-dimensional differential form. In order to connect it with the known three-dimensional objects, a local split of space-time M into space and time is needed. For this purpose, two notions are needed, namely a scalar function σ on M and a family L of curves. The level surfaces of σ , that is, three-dimensional hypersurfaces S (defined by condition $\sigma = t = \text{const}$) are spatial parts of the split. The family L has the following properties: (i) through each point of M one curve κ from L passes, (ii) two different curves do not intersect, (iii) the curves are not tangent to hypersurfaces $\sigma = \text{const}$. These curves are time parts of the local decomposition of the space-time in the following sense

$$T_x(M) = T_x(\kappa) \oplus T_x(S), \quad (1)$$

where T_x denotes the tangent space at point $x \in M$. This decomposition is not equivalent to introducing a metric on M , since no scalar product is introduced on hypersurfaces S . The decomposition (1) allows only to claim that $T_x(S)$ is transversal to $T_x(\kappa)$.¹ The level surfaces of σ determine also a measure of length on curves κ , i.e. a time parameter t .

Let a curve κ be given by the parametrization $x_\kappa(t)$, then one can define vectors $v = dx_\kappa/dt$. Since through each point of M a curve from L passes, we obtain in this manner a vector field v on M . It has the property that the value of the one-form $d\sigma$ on vector v is ²

$$d\sigma(v) = 1. \quad (2)$$

The pair σ, v satisfying (2), after Cruz and Oziewicz [18] can be named *observer field*.

The existence of the one-form $d\sigma$ and the vector field v allows to introduce a $(k-1)$ -form ω_1 and a k -form ω_2 for an arbitrary k -form ω through the formulas

$$\omega_1 = \omega \lrcorner v, \quad \omega_2 = \omega - \omega_1 \wedge d\sigma. \quad (3)$$

The forms ω_i , $i = 1, 2$ are parallel to the vector field v , because $\omega_i \lrcorner v = 0$. The second equation (3) can be rearranged so as to obtain the following expression of the arbitrary k -form ω by ω_i 's:

$$\omega = \omega_1 \wedge d\sigma + \omega_2. \quad (4)$$

The right-hand side of (4) is called by Cruz and Oziewicz [18] the *observer-dependent decomposition* of ω . Hehl and Obukhov [22] call it *1+3 decomposition*. The restrictions of ω_i to the spatial surface S can be called *three-dimensional components* of the four-dimensional form ω .

¹It would be nice if a Minkowski metric on M would be such that $T_x(S)$ is orthogonal to $T_x(\kappa)$. This can be valid only for metrics compatible with the given observer field, but it is not the most general case, see Section 6.

²In this formula and later, d denotes the four-dimensional exterior derivative.

2 Axiom 1: electric charge conservation

The conservation of electric charge was treated as a fundamental law already in the time of Benjamin Franklin. Now one can catch single electrons and protons in traps and count them individually, hence we are sure that the matter carries as a *primary quality* something called *electric charge* which can be counted in principle.

The first axiom is most conveniently formulated in terms of differential forms with the help of the twisted 3-form J of *electrical current density*. Its integral over a 3-dimensional region yields the total charge contained in it which can be determined by counting the charged particles. The conservation of the charge can be described as the differential law:

$$dJ = 0. \quad (5)$$

This law is metric-independent since it is based on the *counting* procedure of elementary charges.

By the prescription (3) we introduce twisted 2-form j and twisted 3-form ρ

$$j = -J \lrcorner v, \quad \rho = J + j \wedge d\sigma. \quad (6)$$

and 1+3 decompose the current density J according to (4)

$$J = -j \wedge d\sigma + \rho \quad (7)$$

By restricting j and ρ to S , one obtains the three-dimensional electric current density and the electric charge density, respectively. The law (5) assumes the familiar shape of the charge continuity equation

$$\dot{\rho} + \underline{d}j = 0, \quad (8)$$

where the dot denotes³ the partial derivative with respect to parameter t and \underline{d} is the three-dimensional exterior derivative acting on differential forms defined on S .

By a theorem of de Rham, equation (5) implies existence of a twisted 2-form G such that

$$J = dG. \quad (9)$$

In this manner the 2-form of electromagnetic excitation is introduced. I propose to call it *Gauss field* in analogy to the name Faraday field for a 2-form F , which will emerge later. This field is not uniquely given by (9), but it can be determined in measurements with the aid of perfect conductors and superconductors. The Gauss field can be 1+3 decomposed with the use of (4):

$$G = -H \wedge d\sigma + D \quad (10)$$

into the twisted 1-form $H = -G \lrcorner v$ of the *magnetic excitation* (traditionally: magnetic field) and the twisted 2-form $D = G + H \wedge d\sigma$ of the *electric excitation* (traditional name: electric displacement or electric induction). Then the relation (9) after restriction to S splits into two three-dimensional inhomogeneous Maxwell equations

$$\underline{d}D = \rho, \quad -\dot{D} + \underline{d}H = j. \quad (11)$$

³It is simultaneously the Lie-Ślebodziński derivative with respect to v , namely $\mathcal{L}_v \rho = d(v \lrcorner \rho) + v \lrcorner (d\rho)$. See also footnote in Section 4.

3 Axiom 2 and 3: Lorentz force density and magnetic flux conservation

Now we would like to introduce another electromagnetic quantity, namely the Faraday field. In analogy to the electrostatic case the simplest four-dimensional definition of the electromagnetic field strength reads:

$$\text{force density} \sim \text{field strength} \times \text{charge current density}.$$

Since the force is a 1-form, the force density should be a 1-form-valued 4-form. After introducing a vector basis e_β of the tangent space, we write down the β -th component f_β of the force density as the 4-form

$$f_\beta = (e_\beta \rfloor F) \wedge J \quad (12)$$

This axiom can be treated as an operational procedure for defining the *electromagnetic field strength* 2-form F , the *Faraday field*. After introducing the 1-form $E = F \rfloor v$ of the *electric field strength* and the 2-form $B = F - E \wedge d\sigma$ of the *magnetic field strength* (traditional name: magnetic induction), the 1+3 decomposition of F reads

$$F = E \wedge d\sigma + B, \quad (13)$$

All experiments done so far attest to the absence of magnetic charge, hence we write the equation

$$dF = 0 \quad (14)$$

expressing the conservation of magnetic flux. The decomposition (13) applied to equation (14) and restriction to S splits it into two three-dimensional homogeneous Maxwell equations:

$$\underline{d}B = 0, \quad \dot{B} + \underline{d}E = j. \quad (15)$$

4 Axiom 4: Energy-momentum density

In electrodynamics, after formulating the Maxwell equations, one has to specify the energy-momentum density of the electromagnetic field. It should be a 1-form-valued 3-form. With the use of the vector basis e_β of the tangent space, we write down the β -th component Σ_β of the energy-momentum density as the 3-form

$$\Sigma_\beta = \frac{1}{2} [F \wedge (e_\beta \rfloor G) - G \wedge (e_\beta \rfloor F)] \quad (16)$$

It can be checked that it has the following relation with the Lorentz force density

$$f_\beta = d\Sigma_\beta + X_\beta, \quad (17)$$

where $X_\beta = \frac{1}{2}(G \wedge \mathcal{L}_\beta \mathcal{F} - \mathcal{F} \wedge \mathcal{L}_\beta G)$ and \mathcal{L}_β is the Lie-Ślebodziński derivative $\mathcal{L}_\beta \phi = d(e_\beta \rfloor \phi) + e_\beta \rfloor (d\phi)$.⁴

⁴Sophus Lie (died in 1899) had not introduced any derivative. This derivative was invented by Władysław Ślebodziński in 1931, see [25]. One year later, van Dantzig [26] quoted the paper by Ślebodziński, but named this notion Lie derivative.

5 Axiom 5: Constitutive relation

Up to now all considerations are generally covariant and metric-free. They are valid in flat Minkowskian as well as in curved pseudo-Riemannian space-time. Therefore, the Maxwell's equations represent the optimal formulation of classical electrodynamics.

The electromagnetical theory in some moment incorporates the metric of a flat or curved space-time via the constitutive relation between the excitation and the field strength. In particular, the standard Maxwell-Lorentz electrodynamics in the vacuum arises when

$$G = \lambda_0 \star F. \quad (18)$$

Here the Hodge star \star is defined by the space-time metric $g_{ij} = \text{diag}(c^2, -1, -1, -1)$, $\lambda_0 = \sqrt{\varepsilon_0/\mu_0}$ is the vacuum admittance.

For a general medium, a local and linear constitutive relation seems to be a reasonable assumption of the axiomatic approach:

$$G_{ij} = \frac{1}{2} \kappa_{ij}{}^{kl} F_{kl}. \quad (19)$$

The constitutive tensor $\kappa_{ij}{}^{kl}$, antisymmetric in pairs ij and kl separately, has 36 independent components. It is useful to decompose it into irreducible parts. Within the premetric framework, the contraction is the only tool for this.

First contraction

$$\kappa_i{}^k = \kappa_{il}{}^{kl} \quad (20)$$

has 16 components. Second contraction

$$\kappa = \kappa_k{}^k = \kappa_{kl}{}^{kl} \quad (21)$$

is the single component. The traceless piece of $\kappa_i{}^k$:

$$\pi_i{}^k = \kappa_i{}^k - \frac{1}{4} \kappa \delta_i^k \quad (22)$$

has 15 components. The κ and π can be immersed in the original constitutive tensor as follows:

$$\kappa_{ij}{}^{kl} = \omega_{ij}{}^{kl} + 2\pi_{[i}{}^{[k} \delta_{j]}^{l]} + \frac{1}{6} \kappa \delta_{[i}^k \delta_{j]}^l \quad (23)$$

ω_{\dots} is the totally traceless part of κ_{\dots} :

$$\omega_{il}{}^{kl} = 0 \quad (24)$$

The right-hand side of (23) is the split of κ_{\dots} according to numbers of components $36 = 20 + 15 + 1$. ω_{\dots} is called *principal part* of the constitutive tensor. π_{\dots} , treated as a function of space-time point, defines a *skewon field*:

$$S_i{}^j = -\frac{1}{2} \pi_i{}^j \quad (25)$$

and κ defines an *axion* field:

$$\alpha = \frac{1}{12} \kappa. \quad (26)$$

The standard Maxwell-Lorentz electrodynamics arises when $S = 0$ and $\alpha = 0$, whereas

$$\omega_{ij}{}^{kl} = \lambda_0 \sqrt{-g} \epsilon_{ijmn} g^{mk} g^{nl}. \quad (27)$$

It has been shown in [23] that when electromagnetic waves in the geometric optics approximation are to exist with the lack of birefringence in vacuum, this implies that there is only one future and only one past directing light cone. As a consequence, the signature of the metric is Lorentzian. The paper [23] has a nice continuation [24].

It is worth to mention that even in flat space-time the spatial part of the metric is not unique – it can be accommodated to a medium, especially when it is non-isotropic. For instance, when a metric is introduced in which the electric permittivity tensor is diagonal, all electrostatic problems can be reduced to that of the isotropic dielectric, see [11]. Similar observation is valid for the anisotropic magnetic medium and the magnetostatics.

Recently an exotic medium was considered for which the principal and skewon parts of the constitutive tensor are zero and only axion part remains. In this case the constitutive relation assumes the simplest form

$$G = \alpha F \quad (28)$$

with the pseudoscalar α . Lindell and Sihvola [27] called it *perfect electromagnetic conductor (PEMC)*, because the limit $\alpha \rightarrow 0$ yields the perfect magnetic conductor, and the limit $1/\alpha \rightarrow 0$ gives the perfect electric conductor. The proportionality relation (28) implies that the energy-momentum density (16) is zero for the electromagnetic field present in such a medium. This means that when electromagnetic wave enters it, the wave can not transmit energy. I have considered [28] plane electromagnetic wave incident normally from an ordinary medium on a slab made of PEMC. The boundary conditions on the front and back interface between the PEMC and ordinary media on two sides imply that the electromagnetic wave does not enter the second medium behind the slab.

There is also another axiom in the book of Hehl and Obukhov, namely that concerning the splitting of the electromagnetic quantities in the continuous media into external and matter parts, but I shall not dwell on it.

6 Inertial versus non-inertial observers

An interesting direction of research concerns the reference frames and observers, see Refs. [29], [30], [31], [32], [18], [22], [33]. There is no established terminology what is an observer and what is a reference frame. For instance W.A. Rodrigues and E.C. de Oliveira [33] define reference frame as as a (time-like) vector field and observers as integral lines of the vector field. F.W. Hehl and Yu.N. Obukhov [22] use exchangeably coordinate frame and frame of reference. Most often an inertial

reference frame is defined with the aid of connection, see [29], [30] and [33] Section 5.1.1. J.J. Cruz and Z. Oziewicz [18] (see also [31], [32]) consider more general notion of *observer field*, when the pair: one-form $d\sigma$ and vector field v of the Introduction are replaced by a (1,1)-tensor field $p = v \otimes d\sigma$ which in each point of M is a projection of arbitrary tensors onto time axis of the observer. Then they ponder on how one can discriminate between inertial and non-inertial observers. Cruz and Oziewicz postulate that it is possible to define an inertial observer without connection if one introduces so called Frölicher-Nijenhuis operation on differential forms and Frobenius algebra of massive observers. Then the observer is inertial when the vector field v is divergence-free and acceleration of the observer vanishes. (For the exact meaning of these terms see [18].)

The named authors consider also another generalization, namely that the spaces $T_x(\kappa)$ and $T_x(S)$ are not orthogonal. This means that the decomposition (1) is not compatible with the Minkowski-Riemann metric g in M . In this case Cruz and Oziewicz say that the observer field p is *not g -orthogonal*. They are also able to write the charge continuity equation (8) and the four Maxwell equations (11, 15) not by the restriction to spatial surfaces S , but as equations for bona fide four-dimensional observer-dependent differential forms j , ρ , E , B , H , and D , and these equations are observer-dependent, but still pre-metric.

Afterwards, the observer-dependent vector fields \mathbf{B} , \mathbf{E} , \mathbf{D} , \mathbf{H} are introduced in the presence of the space-time metric g and the Maxwell equations are written for them. In the equations, extra terms are present when the observer field is non-inertial. These terms depend on acceleration and rotary motion of the observers. Some of these formulas were derived already by Jerzy Kocik ([32]). The modified Maxwell equations are displayed also in the book by Hehl and Obukhov, [22] Section B.4.4, without introducing the vector fields. The additional terms appear when the components of the electromagnetic field are expressed in terms of the coframe of the non-inertial reference frame. It is interesting to know about possible observability of the extra terms. The Earth is an example of non-inertial system of reference, hence such terms should be present in the Maxwell equations in this system. A private message from Zbigniew Oziewicz: the magnitude of them is too small to be detectable now.

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